# 2022 FALL REAL ANALYSIS (I) @ NYCU APPL. MATH.
# HOMEWORK 2

- Please answer the following questions in details, which means you need to state all theorems and all reasons you have been using.
- Please mark your name, student ID, and question numbers clearly on your answer sheet. The deadline to hand in the exercise is on October 7, 2022.

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(1) Show that the Borel $\sigma$-algebra $\mathcal{B}$ in $\mathbb{R}^n$ is the smallest $\sigma$-algebra containing the closed sets in $\mathbb{R}^n$.

(2) Let $E \subset \mathbb{R}^n$ be any subset, recalling that the outer measure is defined via

$$m_*(E) = \inf \{ m(\mathcal{O}) : \ E \subset \mathcal{O}, \text{ and } \mathcal{O} \text{ is an open set in } \mathbb{R}^n \} .$$

One can also define an inner measure $m^*(E)$ by

$$m^*(E) = \sup \{ m(F) : \ F \subset E, \text{ and } F \text{ is a closed set in } \mathbb{R}^n \} .$$

Show that

(a) $m^*(E) \leq m_*(E)$.

(b) $E$ is measurable if and only if that $m^*(E) = m_*(E)$, provided that $m_*(E) < \infty$.

(3) We have introduced $F_\sigma$ and $G_\delta$ sets in $\mathbb{R}^n$.

(a) Show that a closed set is $G_\delta$ but an open set is $F_\sigma$.

(b) Construct a set, which is $F_\sigma$ but not $G_\delta$.

(c) Construct a Borel set, which is neither $G_\delta$ nor $F_\sigma$.

(4) Construct an example which is a Lebesgue measurable set but not a Borel measurable set.

(5) Suppose $E$ is a measure zero set in $\mathbb{R}$. Show that there exists a sequence of open sets $\{\mathcal{O}_n\}$ such that $E \subset \bigcap_{n=1}^{\infty} \mathcal{O}_n$ and $\lim_{n\to\infty} m(\mathcal{O}_n) = 0$.

(6) Try to understand what is a non-measurable set (in the Lebesgue sense)[1].

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[1]Please do not hand in this problem to your TA, and the construction of non-measurable sets can be found in any references.